Eighth Edition

MATHEMATICS

FOR ECONOMICS AND BUSINESS

IAN JACQUES

ALWAYS LEARNING

PEARSON

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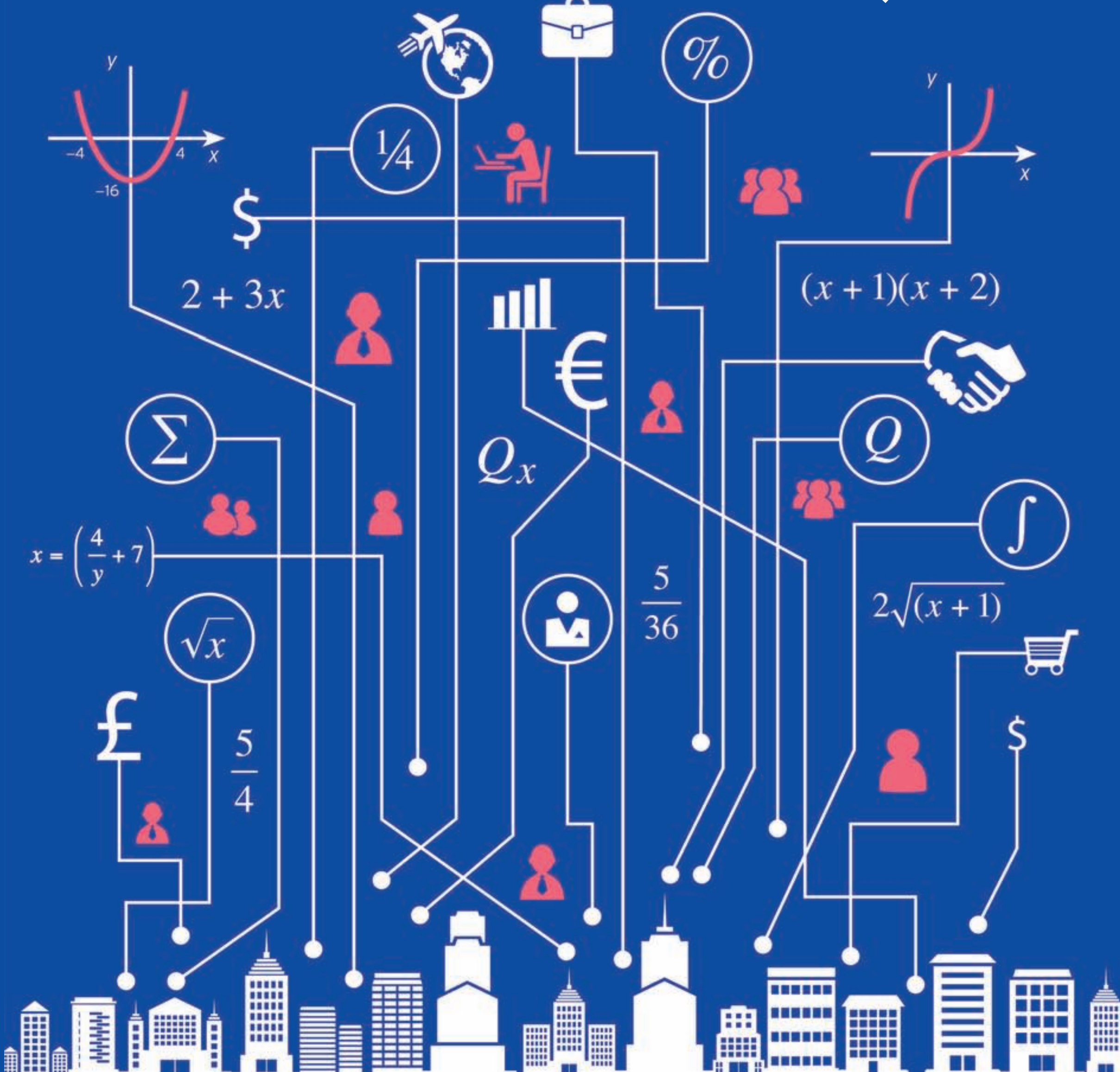
IAN JACQUES



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To Victoria, Lewis and Celia

CONTENTS

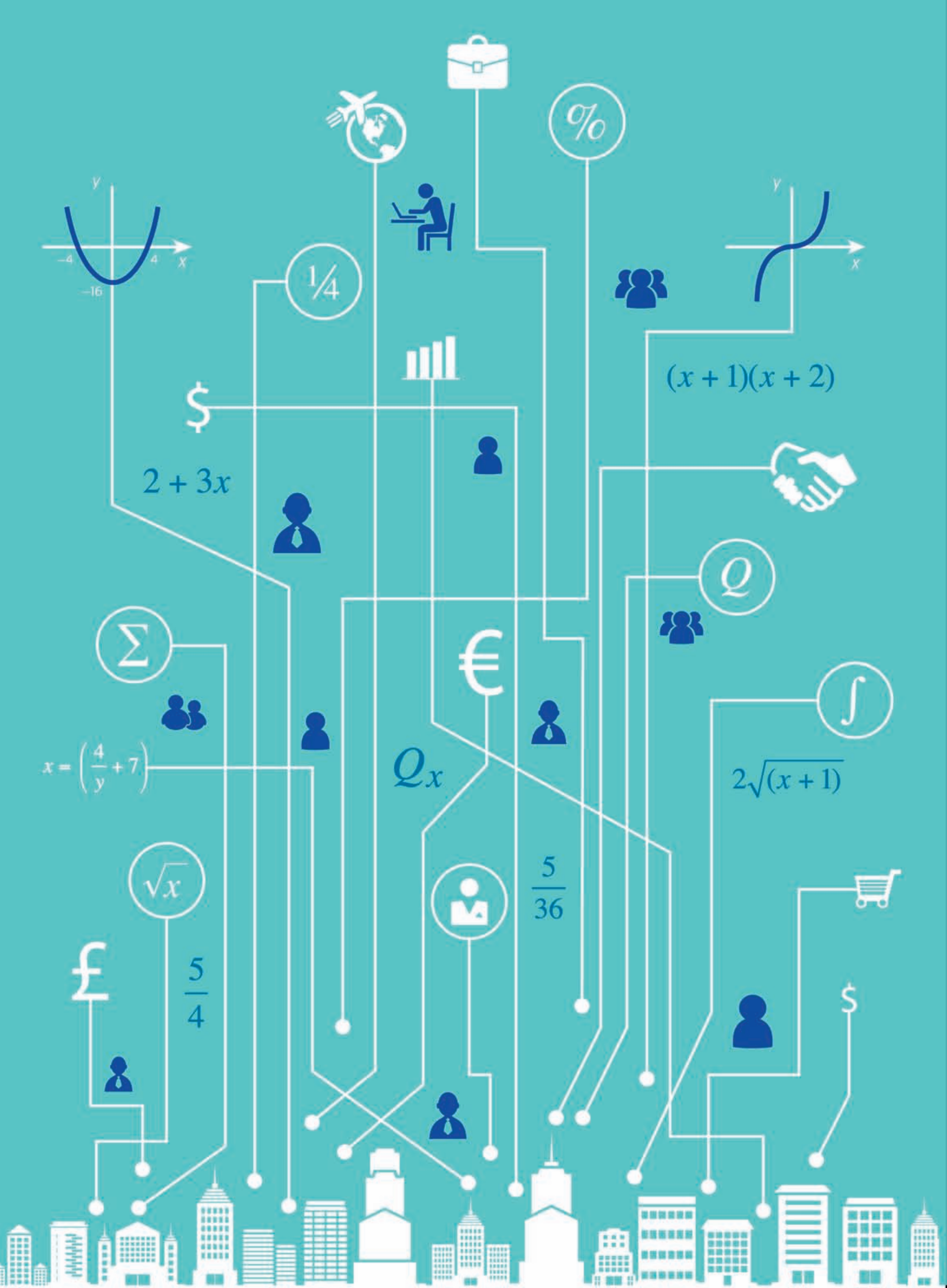
PREFACE

This book is intended primarily for students on economics, business studies and management courses. It assumes very little prerequisite knowledge, so it can be read by students who have not undertaken a mathematics course for some time. The style is informal and the book contains a large number of worked examples. Students are encouraged to tackle problems for themselves as they read through each section. Detailed solutions are provided so that all answers can be checked. Consequently, it should be possible to work through this book on a self-study basis. The material is wide ranging, and varies from elementary topics such as percentages and linear equations to more sophisticated topics such as constrained optimisation of multivariate functions. The book should therefore be suitable for use on both low- and high-level quantitative methods courses.

This book was first published in 1991. The prime motivation for writing it then was to try to produce a textbook that students could actually read and understand for themselves. This remains the guiding principle when writing this eighth edition. There are two significant improvements based on suggestions made from many anonymous reviewers of previous editions (thank you).

- More worked examples and problems related to business have been included.
- Additional questions have been included in the core exercises and more challenging problems are available in the starred exercises.

Ian Jacques



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INTRODUCTION

Getting Started

NOTES FOR STUDENTS: HOW TO USE THIS BOOK

I am always amazed by the mix of students on first-year economics courses. Some have not acquired any mathematical knowledge beyond elementary algebra (and even that can be of a rather dubious nature), some have never studied economics before in their lives, while others have passed preliminary courses in both. Whatever category you are in, I hope that you will find this book of value. The chapters covering algebraic manipulation, simple calculus, finance, matrices and linear programming should also benefit students on business studies and management courses.

The first few chapters are aimed at complete beginners and students who have not taken mathematics courses for some time. I would like to think that these students once enjoyed mathematics and had every intention of continuing their studies in this area, but somehow never found the time to fit it into an already overcrowded academic timetable. However, I suspect that the reality is rather different. Possibly they hated the subject, could not understand it and dropped it at the earliest opportunity. If you find yourself in this position, you are probably horrified to discover that you must embark on a quantitative methods course with an examination looming on the horizon. However, there is no need to worry. My experience is that every student is capable of passing a mathematics examination. All that is required is a commitment to study and a willingness to suspend any prejudices about the subject gained at school. The fact that you have bothered to buy this book at all suggests that you are prepared to do both.

To help you get the most out of this book, let me compare the working practices of economics and engineering students. The former rarely read individual books in any great depth. They tend to visit college libraries (usually several days after an essay was due to be handed in) and skim through a large number of books, picking out the relevant information. Indeed, the ability to read selectively and to compare various sources of information is an important skill that all arts and social science students must acquire. Engineering students, on the other hand, are more likely to read just a few books in any one year. They read each of these from cover to cover and attempt virtually every problem en route. Even though you are most definitely not an engineer, it is the engineering approach that you need to adopt while studying mathematics. There are several reasons for this. Firstly, a mathematics book can never be described, even by its most ardent admirers, as a good bedtime read. It can take an hour or two of concentrated effort to understand just a few pages of a mathematics text. You are therefore recommended to work through this book systematically in short bursts rather than to attempt to read whole chapters. Each section is designed to take between one and two hours to complete and this is quite sufficient for a single session. Secondly, mathematics is a hierarchical subject in which one topic follows on from the next. A construction firm building an office block is hardly likely to erect the fiftieth storey without making sure that the intermediate floors and foundations are securely in place. Likewise, you cannot 'dip' into the middle of a mathematics book and expect to follow it unless you have satisfied the prerequisites for that topic. Finally, you actually need to do mathematics yourself before you can understand it. No matter how wonderful your lecturer is, and no matter how many problems are discussed in class, it is only by solving problems yourself that you are ever going to become confident in using and applying mathematical techniques. For this reason, several problems are interspersed within the text and you are encouraged to tackle these as you go along. You will require writing paper, graph paper, pens and a calculator for this. There is no need to buy an expensive calculator unless you are feeling particularly wealthy at the moment. A bottom-of-the-range **scientific** calculator should be good enough. Answers to every question are printed at the back of this book so that you can check your own answers quickly as you go along. However, please avoid the temptation to look at them until you have made an honest attempt at each one. Remember that in the future you may well have to sit down in an uncomfortable chair, in front of a blank sheet of paper, and be expected to produce solutions to examination questions of a similar type.

At the end of each section there are two parallel exercises. The non-starred exercises are intended for students who are meeting these topics for the first time and the questions are designed to consolidate basic principles. The starred exercises are more challenging but still cover the full range so that students with greater experience will be able to concentrate their efforts on these questions without having to pick-and-mix from both exercises. The chapter dependence is shown in Figure I.1. If you have studied some advanced mathematics before, you will discover that parts of Chapters 1, 2 and 4 are familiar. However, you may find that the sections on economics applications contain new material. You are best advised to test yourself by attempting a selection of problems from the starred exercise in each section to see if you need to read through it as part of a refresher course. Economics students in a desperate hurry to experience the delights of calculus can miss out Chapter 3 without any loss of continuity and move straight on to Chapter 4. The mathematics of finance is probably more relevant to business and accountancy students, although you can always read it later if it is part of your economics syllabus.

I hope that this book helps you to succeed in your mathematics course. You never know, you might even enjoy it. Remember to wear your engineer's hat while reading the book. I have done my best to make the material as accessible as possible. The rest is up to you!

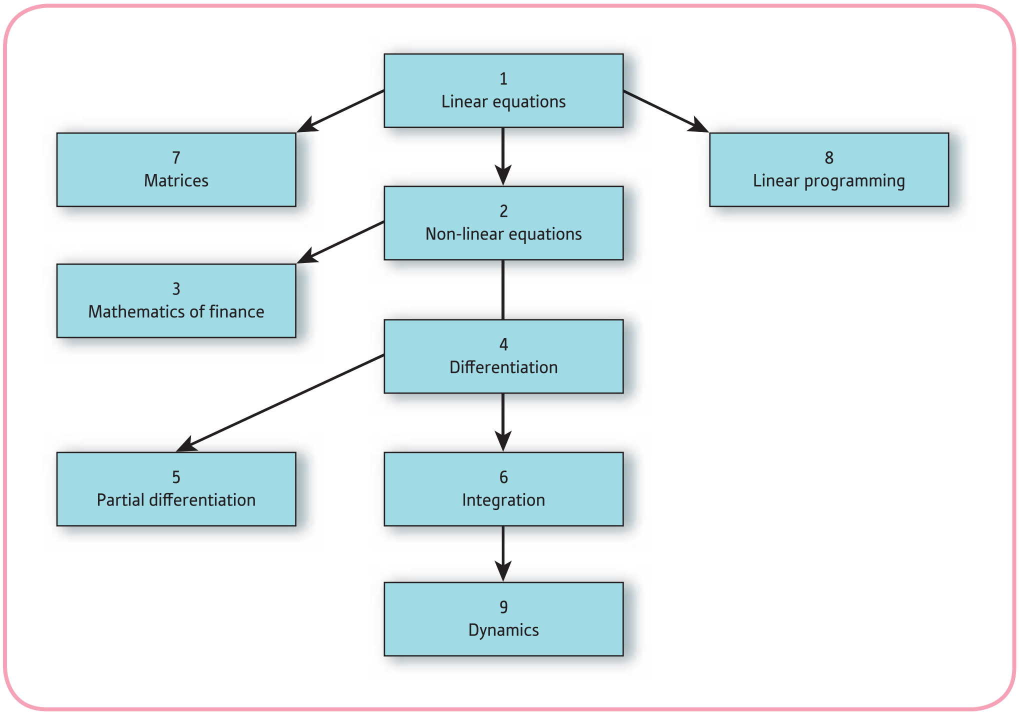
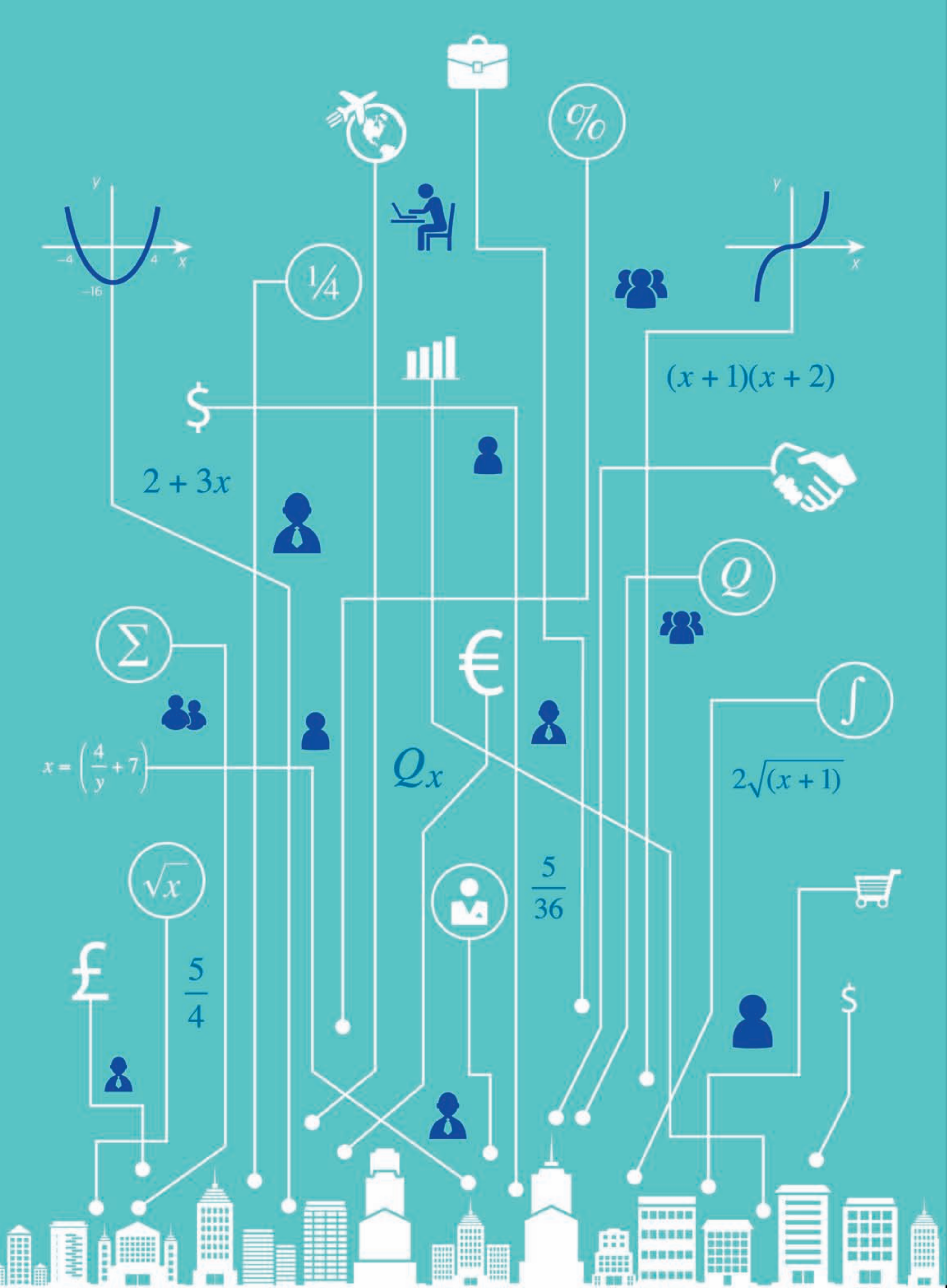


Figure I.1



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CHAPTER 1
Linear Equations

The main aim of this chapter is to introduce the mathematics of linear equations. This is an obvious first choice in an introductory text, since it is an easy topic which has many applications. There are seven sections, which are intended to be read in the order that they appear.

Sections 1.1, 1.2, 1.3, 1.4 and 1.6 are devoted to mathematical methods. They serve to revise the rules of arithmetic and algebra, which you probably met at school but may have forgotten. In particular, the properties of negative numbers and fractions are considered. A reminder is given on how to multiply out brackets and how to manipulate mathematical expressions. You are also shown how to solve simultaneous linear equations. Systems of two equations in two unknowns can be solved using graphs, which are described in Section 1.3. However, the preferred method uses elimination, which is considered in Section 1.4. This algebraic approach has the advantage that it always gives an exact solution and it extends readily to larger systems of equations.

The remaining two sections are reserved for applications in microeconomics and macroeconomics. You may be pleasantly surprised by how much economic theory you can analyse using just the basic mathematical tools considered here. Section 1.5 introduces the fundamental concept of an economic function and describes how to calculate equilibrium prices and quantities in supply and demand theory. Section 1.7 deals with national income determination in simple macroeconomic models.

The first six sections underpin the rest of the book and are essential reading. The final section is not quite as important and can be omitted at this stage.

SECTION 1.1

Introduction to algebra

Objectives

At the end of this section you should be able to:

- Add, subtract, multiply and divide negative numbers.
- Understand what is meant by an algebraic expression.
- Evaluate algebraic expressions numerically.
- Simplify algebraic expressions by collecting like terms.
- Multiply out brackets.
- Factorise algebraic expressions.

ALGEBRA IS BORING

There is no getting away from the fact that algebra *is* boring. Doubtless there are a few enthusiasts who get a kick out of algebraic manipulation, but economics and business students are rarely to be found in this category. Indeed, the mere mention of the word 'algebra' is enough to strike fear into the heart of many a first-year student. Unfortunately, you cannot get very far with mathematics unless you have completely mastered this topic. An apposite analogy is the game of chess. Before you can begin to play a game of chess it is necessary to go through the tedium of learning the moves of individual pieces. In the same way it is essential that you learn the rules of algebra before you can enjoy the 'game' of mathematics. Of course, just because you know the rules does not mean that you are going to excel at the game and no one is expecting you to become a grandmaster of mathematics. However, you should at least be able to follow the mathematics presented in economics books and journals, as well as being able to solve simple problems for yourself.

Advice

If you have studied mathematics recently then you will find the material in the first few sections of the book fairly straightforward. You may prefer just to try the questions in the starred exercise at the end of each section to get yourself back up to speed. However, if it has been some time since you have studied this subject our advice is very different. Please work through the material thoroughly even if it is vaguely familiar. Make sure that you do the problems as they arise, checking your answers with those provided at the back of this book. The material has been broken down into three subsections:

- negative numbers
- expressions
- brackets.

You might like to work through these subsections on separate occasions to enable the ideas to sink in. To rush this topic now is likely to give you only a half-baked understanding, which will result in hours of frustration when you study the later chapters of this book.

1.1.1 Negative numbers

In mathematics numbers are classified into one of three types: positive, negative or zero. At school you were probably introduced to the idea of a negative number via the temperature on a thermometer scale measured in degrees centigrade. A number such as −5 would then be interpreted as a temperature of 5 degrees below freezing. In personal finance a negative bank balance would indicate that an account is 'in the red' or 'in debit'. Similarly, a firm's profit of −500 000 signifies a loss of half a million.

The rules for the multiplication of negative numbers are

negative × negative = positive

negative × positive = negative

It does not matter in which order two numbers are multiplied, so

positive × negative = negative

These rules produce, respectively,

$$(-2) \times (-3) = 6$$
$$(-4) \times 5 = -20$$
$$7 \times (-5) = -35$$

Also, because division is the same sort of operation as multiplication (it just undoes the result of multiplication and takes you back to where you started), exactly the same rules apply when one number is divided by another. For example,

$$(-15) \div (-3) = 5$$
$$(-16) \div 2 = -8$$
$$2 \div (-4) = -1/2$$

In general, to multiply or divide lots of numbers it is probably simplest to ignore the signs to begin with and just to work the answer out. The final result is negative if the total number of minus signs is odd and positive if the total number is even.

Example

Evaluate

(a) $(-2) \times (-4) \times (-1) \times 2 \times (-1) \times (-3)$ **(b)** $\dfrac{5 \times (-4) \times (-1) \times (-3)}{(-6) \times 2}$

Solution

(a) Ignoring the signs gives

$$2 \times 4 \times 1 \times 2 \times 1 \times 3 = 48$$

There are an odd number of minus signs (in fact, five) so the answer is −48.

(b) Ignoring the signs gives

$$\frac{5 \times 4 \times 1 \times 3}{6 \times 2} = \frac{60}{12} = 5$$

There are an even number of minus signs (in fact, four) so the answer is 5.

Advice

Attempt the following problem yourself both with and without a calculator. On most machines a negative number such as –6 is entered by pressing the button labelled (–) followed by 6.

Practice Problem

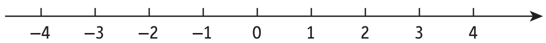
1. **(1)** Without using a calculator evaluate

(a) $5 \times (-6)$ **(b)** $(-1) \times (-2)$ **(c)** $(-50) \div 10$

(d) $(-5) \div (-1)$ **(e)** $2 \times (-1) \times (-3) \times 6$ **(f)** $\dfrac{2 \times (-1) \times (-3) \times 6}{(-2) \times 3 \times 6}$

(2) Confirm your answer to part (1) using a calculator.

To add or subtract negative numbers it helps to think in terms of a number line:



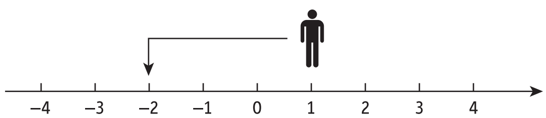
If b is a positive number then

$a - b$

can be thought of as an instruction to start at a and to move b units to the left. For example,

$1 - 3 = -2$

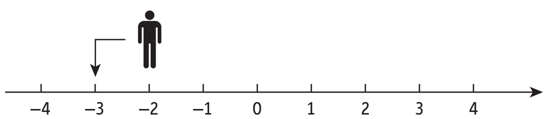
because if you start at 1 and move 3 units to the left, you end up at –2:



Similarly,

$-2 - 1 = -3$

because 1 unit to the left of –2 is –3.



On the other hand,

$a - (-b)$

is taken to be $a + b$. This follows from the rule for multiplying two negative numbers, since

$-(-b) = (-1) \times (-b) = b$

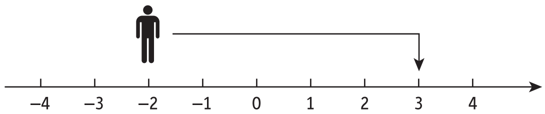
Consequently, to evaluate

$a - (-b)$

you start at a and move b units to the right (that is, in the positive direction). For example,

$$-2 - (-5) = -2 + 5 = 3$$

because if you start at −2 and move 5 units to the right you end up at 3.



Practice Problem

2. **(1)** Without using a calculator evaluate

(a) $1 - 2$ **(b)** $-3 - 4$ **(c)** $1 - (-4)$

(d) $-1 - (-1)$ **(e)** $-72 - 19$ **(f)** $-53 - (-48)$

(2) Confirm your answer to part (1) using a calculator.

1.1.2 Expressions

In algebra letters are used to represent numbers. In pure mathematics the most common letters used are x and y. However, in applications it is helpful to choose letters that are more meaningful, so we might use Q for quantity and I for investment. An algebraic expression is then simply a combination of these letters, brackets and other mathematical symbols such as + or −. For example, the expression

$$P\left(1+\frac{r}{100}\right)^n$$

can be used to work out how money in a savings account grows over a period of time. The letters P, r and n represent the original sum invested (called the principal – hence the use of the letter P), the rate of interest and the number of years, respectively. To work it all out, you not only need to replace these letters by actual numbers, but you also need to understand the various conventions that go with algebraic expressions such as this.

In algebra when we multiply two numbers represented by letters we usually suppress the multiplication sign between them. The product of a and b would simply be written as ab without bothering to put the multiplication sign between the symbols. Likewise when a number represented by the letter Y is doubled we write $2Y$. In this case we not only suppress the multiplication sign but adopt the convention of writing the number in front of the letter. Here are some further examples:

$P \times Q$ is written as PQ

$d \times 8$ is written as $8d$

$n \times 6 \times t$ is written as $6nt$

$z \times z$ is written as z^2 (using the index 2 to indicate squaring a number)

$1 \times t$ is written as t (since multiplying by 1 does not change a number)

In order to evaluate these expressions it is necessary to be given the numerical value of each letter. Once this has been done you can work out the final value by performing the operations in the following order:

Brackets first	(B)
Indices second	(I)
Division and Multiplication third	(DM)
Addition and Subtraction fourth	(AS)

This is sometimes remembered using the acronym BIDMAS and it is essential that this ordering is used for working out all mathematical calculations. For example, suppose you wish to evaluate each of the following expressions when $n = 3$:

$2n^2$ and $(2n)^2$

Substituting $n = 3$ into the first expression gives

$2n^2 = 2 \times 3^2$ (the multiplication sign is revealed when we switch from algebra to numbers)

$= 2 \times 9$ (according to BIDMAS indices are worked out before multiplication)

$= 18$

whereas in the second expression we get

$(2n)^2 = (2 \times 3)^2$ (again the multiplication sign is revealed)

$= 6^2$ (according to BIDMAS we evaluate the inside of the brackets first)

$= 36$

The two answers are not the same so the order indicated by BIDMAS really does matter. Looking at the previous list, notice that there is a tie between multiplication and division for third place, and another tie between addition and subtraction for fourth place. These pairs of operations have equal priority and under these circumstances you work from left to right when evaluating expressions. For example, substituting $x = 5$ and $y = 4$ in the expression, $x - y + 2$, gives

$x - y + 2 = 5 - 4 + 2$

$= 1 + 2$ (reading from left to right, subtraction comes first)

$= 3$

Example

(a) Find the value of $2x - 3y$ when $x = 9$ and $y = 4$.

(b) Find the value of $2Q^2 + 4Q + 150$ when $Q = 10$.

(c) Find the value of $5a - 2b + c$ when $a = 4$, $b = 6$ and $c = 1$.

(d) Find the value of $(12 - t) - (t - 1)$ when $t = 4$.

Solution

(a) $2x - 3y = 2 \times 9 - 3 \times 4$ (substituting numbers)

$= 18 - 12$ (multiplication has priority over subtraction)

$= 6$

(b) $2Q^2 + 4Q + 150 = 2 \times 10^2 + 4 \times 10 + 150$ (substituting numbers)

$= 2 \times 100 + 4 \times 10 + 150$ (indices have priority over multiplication and addition)

$= 200 + 40 + 150$ (multiplication has priority over addition)

$= 390$

(c) $5a - 2b + c = 5 \times 4 - 2 \times 6 + 1$ (substituting numbers)

$= 20 - 12 + 1$ (multiplication has priority over addition and subtraction)

$= 8 + 1$ (addition and subtraction have equal priority, so work from left to right)

$= 9$

(d) $(12 - t) - (t - 1) = (12 - 4) - (4 - 1)$ (substituting numbers)

$= 8 - 3$ (brackets first)

$= 5$

Practice Problem

3. Evaluate each of the following by replacing the letters by the given numbers:

(a) $2Q + 5$ when $Q = 7$.

(b) $5x^2y$ when $x = 10$ and $y = 3$.

(c) $4d - 3f + 2g$ when $d = 7$, $f = 2$ and $g = 5$.

(d) $a(b + 2c)$ when $a = 5$, $b = 1$ and $c = 3$.

Like terms are multiples of the same letter (or letters). For example, $2P$, $-34P$ and $0.3P$ are all multiples of P and so are like terms. In the same way, xy, $4xy$ and $69xy$ are all multiples of xy and so are like terms. If an algebraic expression contains like terms which are added or subtracted together then it can be simplified to produce an equivalent shorter expression.

Example

Simplify each of the following expressions (where possible):

(a) $2a + 5a - 3a$

(b) $4P - 2Q$

(c) $3w + 9w^2 + 2w$

(d) $3xy + 2y^2 + 9x + 4xy - 8x$

Solution

(a) All three are like terms since they are all multiples of a so the expression can be simplified:

$$2a + 5a - 3a = 4a$$

(b) The terms $4P$ and $2Q$ are unlike because one is a multiple of P and the other is a multiple of Q so the expression cannot be simplified.

(c) The first and last are like terms since they are both multiples of w so we can collect these together and write

$$3w + 9w^2 + 2w = 5w + 9w^2$$

This cannot be simpified any further because $5w$ and $9w^2$ are unlike terms.

(d) The terms $3xy$ and $4xy$ are like terms, and $9x$ and $8x$ are also like terms. These pairs can therefore be collected together to give

$$3xy + 2y^2 + 9x + 4xy - 8x = 7xy + 2y^2 + x$$

Notice that we write just x instead of $1x$ and also that no further simplication is possible since the final answer involves three unlike terms.

Practice Problem

4. Simplify each of the following expressions, where possible:

(a) $2x + 6y - x + 3y$ **(b)** $5x + 2y - 5x + 4z$ **(c)** $4Y^2 + 3Y - 43$

(d) $8r^2 + 4s - 6rs - 3s - 3s^2 + 7rs$ **(e)** $2e^2 + 5f - 2e^2 - 9f$ **(f)** $3w + 6W$

(g) $ab - ba$

1.1.3 Brackets

It is useful to be able to take an expression containing brackets and rewrite it as an equivalent expression without brackets and vice versa. The process of removing brackets is called 'expanding brackets' or 'multiplying out brackets'. This is based on the **distributive law**, which states that for any three numbers a, b and c

$$a(b + c) = ab + ac$$

It is easy to verify this law in simple cases. For example, if $a = 2$, $b = 3$ and $c = 4$ then the left-hand side is

$$2(3 + 4) = 2 \times 7 = 14$$

However,

$$ab = 2 \times 3 = 6 \quad \text{and} \quad ac = 2 \times 4 = 8$$

and so the right-hand side is $6 + 8$, which is also 14.

This law can be used when there are any number of terms inside the brackets. We have

$$a(b + c + d) = ab + ac + ad$$
$$a(b + c + d + e) = ab + ac + ad + ae$$

and so on.

It does not matter in which order two numbers are multiplied, so we also have

$$(b + c)a = ba + ca$$
$$(b + c + d)a = ba + ca + da$$
$$(b + c + d + e)a = ba + ca + da + ea$$

Example

Multiply out the brackets in

(a) $x(x - 2)$

(b) $2(x + y - z) + 3(z + y)$

(c) $x + 3y - (2y + x)$

Solution

(a) The use of the distributive law to multiply out $x(x - 2)$ is straightforward. The x outside the bracket multiplies the x inside to give x^2. The x outside the bracket also multiplies the -2 inside to give $-2x$. Hence

$$x(x - 2) = x^2 - 2x$$

(b) To expand

$$2(x + y - z) + 3(z + y)$$

we need to apply the distributive law twice. We have

$$2(x + y - z) = 2x + 2y - 2z$$
$$3(z + y) = 3z + 3y$$

Adding together gives

$$\begin{aligned} 2(x + y - z) + 3(z + y) &= 2x + 2y - 2z + 3z + 3y \\ &= 2x + 5y + z \qquad \text{(collecting like terms)} \end{aligned}$$

(c) It may not be immediately apparent how to expand

$$x + 3y - (2y + x)$$

However, note that

$$-(2y + x)$$

is the same as

$$(-1)(2y + x)$$

which expands to give

$$(-1)(2y) + (-1)x = -2y - x$$

Hence

$$x + 3y - (2y + x) = x + 3y - 2y - x = y$$

after collecting like terms.

Advice

In this example the solutions are written out in painstaking detail. This is done to show you precisely how the distributive law is applied. The solutions to all three parts could have been written down in only one or two steps of working. You are, of course, at liberty to compress the working in your own solutions, but please do not be tempted to overdo this. You might want to check your answers at a later date and may find it difficult if you have tried to be too clever.

Practice Problem

5. Multiply out the brackets, simplifying your answer as far as possible.

(a) $(5 - 2z)z$ **(b)** $6(x - y) + 3(y - 2x)$ **(c)** $x - y + z - (x^2 + x - y)$

Mathematical formulae provide a precise way of representing calculations that need to be worked out in many business models. However, it is important to realise that these formulae may only be valid for a restricted range of values. Most large companies have a policy to reimburse employees for use of their cars for travel: for the first 50 miles they may be able to claim 90 cents a mile but this could fall to 60 cents a mile thereafter. If the distance, x miles, is no more than 50 miles then travel expenses, E, (in dollars) could be worked out using formula, $E = 0.9x$. If x exceeds 50 miles the employee can claim \$0.90 a mile for the first 50 miles but only \$0.60 a mile for the last $(x - 50)$ miles. The total amount is then

$$\begin{aligned} E &= 0.9 \times 50 + 0.6(x - 50) \\ &= 45 + 0.6x - 30 \\ &= 15 + 0.6x \end{aligned}$$

Travel expenses can therefore be worked out using two separate formulae:

- $E = 0.9x$ when x is no more than 50 miles
- $E = 15 + 0.6x$ when x exceeds 50 miles.

Before we leave this topic a word of warning is in order. Be careful when removing brackets from very simple expressions such as those considered in part (c) in the previous worked example and practice problem. A common mistake is to write

$(a + b) - (c + d) = a + b - c + d$ This is NOT true

The distributive law tells us that the -1 multiplying the second bracket applies to the d as well as the c so the correct answer has to be

$$(a + b) - (c + d) = a + b - c - d$$

In algebra, it is sometimes useful to reverse the procedure and put the brackets back in. This is called **factorisation**. Consider the expression $12a + 8b$. There are many numbers which divide into both 8 and 12. However, we always choose the biggest number, which is 4 in this case, so we attempt to take the factor of 4 outside the brackets:

$$12a + 8b = 4(? + ?)$$

where the ? indicate some mystery terms inside the brackets. We would like 4 multiplied by the first term in the brackets to be $12a$ so we are missing $3a$. Likewise if we are to generate an $8b$ the second term in the brackets will have to be $2b$.

Hence

$$12a + 8b = 4(3a + 2b)$$

As a check, notice that when you expand the brackets on the right-hand side you really do get the expression on the left-hand side.

Example

Factorise

(a) $6L - 3L^2$

(b) $5a - 10b + 20c$

Solution

(a) Both terms have a common factor of 3. Also, because $L^2 = L \times L$, both $6L$ and $-3L^2$ have a factor of L. Hence we can take out a common factor of $3L$ altogether.

$$6L - 3L^2 = 3L(2) - 3L(L) = 3L(2 - L)$$

(b) All three terms have a common factor of 5 so we write

$$5a - 10b + 20c = 5(a) - 5(2b) + 5(4c) = 5(a - 2b + 4c)$$

Practice Problem

6. Factorise

(a) $7d + 21$ (b) $16w - 20q$ (c) $6x - 3y + 9z$ (d) $5Q - 10Q^2$

We conclude our discussion of brackets by describing how to multiply two brackets together. In the expression $(a + b)(c + d)$ the two terms a and b must each multiply the single bracket $(c + d)$ so

$$(a + b)(c + d) = a(c + d) + b(c + d)$$

The first term $a(c + d)$ can itself be expanded as $ac + ad$. Likewise, $b(c + d) = bc + bd$. Hence

$$(a + b)(c + d) = ac + ad + bc + bd$$

This procedure then extends to brackets with more than two terms:

$$(a + b)(c + d + e) = a(c + d + e) + b(c + d + e) = ac + ad + ae + bc + bd + be$$

Example

Multiply out the brackets

(a) $(x + 1)(x + 2)$ **(b)** $(x + 5)(x - 5)$ **(c)** $(2x - y)(x + y - 6)$

simplifying your answer as far as possible.

Solution

(a)
$$\begin{aligned}(x + 1)(x + 2) &= x(x + 2) + (1)(x + 2)\\ &= x^2 + 2x + x + 2\\ &= x^2 + 3x + 2\end{aligned}$$

(b)
$$\begin{aligned}(x + 5)(x - 5) &= x(x - 5) + 5(x - 5)\\ &= x^2 - 5x + 5x - 25\\ &= x^2 - 25\end{aligned}$$

the *x*s cancel

(c)
$$\begin{aligned}(2x - y)(x + y - 6) &= 2x(x + y - 6) - y(x + y - 6)\\ &= 2x^2 + 2xy - 12x - yx - y^2 + 6y\\ &= 2x^2 + xy - 12x - y^2 + 6y\end{aligned}$$

Practice Problem

7. Multiply out the brackets.

(a) $(x + 3)(x - 2)$

(b) $(x + y)(x - y)$

(c) $(x + y)(x + y)$

(d) $(5x + 2y)(x - y + 1)$

Looking back at part (b) of the previous worked example, notice that

$$(x + 5)(x - 5) = x^2 - 25 = x^2 - 5^2$$

Quite generally

$$\begin{aligned}(a + b)(a - b) &= a(a - b) + b(a - b)\\ &= a^2 - ab + ba - b^2\\ &= a^2 - b^2\end{aligned}$$

The result

$$\boxed{a^2 - b^2 = (a + b)(a - b)}$$

is called the **difference of two squares** formula. It provides a quick way of factorising certain expressions.

Example

Factorise the following expressions:

(a) $x^2 - 16$ **(b)** $9x^2 - 100$

Solution

(a) Noting that

$$x^2 - 16 = x^2 - 4^2$$

we can use the difference of two squares formula to deduce that

$$x^2 - 16 = (x + 4)(x - 4)$$

(b) Noting that

$$9x^2 - 100 = (3x)^2 - (10)^2$$

$(3x)^2 = 3x \times 3x$
$= 9x^2$

we can use the difference of two squares formula to deduce that

$$9x^2 - 100 = (3x + 10)(3x - 10)$$

Practice Problem

8. Factorise the following expressions:

(a) $x^2 - 64$ **(b)** $4x^2 - 81$

Advice

This completes your first piece of mathematics. We hope that you have not found it quite as bad as you first thought. There now follow a few extra problems to give you more practice. Not only will they help to strengthen your mathematical skills, but also they should improve your overall confidence. There are two alternative exercises available. Exercise 1.1 is suitable for students whose mathematics may be rusty and who need to consolidate their understanding. Exercise 1.1* contains more challenging problems and so is more suitable for those students who have found this section very easy.

Key Terms

Difference of two squares The algebraic result which states that $a^2 - b^2 = (a + b)(a - b)$.

Distributive law The law of arithmetic which states that $a(b + c) = ab + ac$ for any numbers, a, b, c.

Factorisation The process of writing an expression as a product of simpler expressions using brackets.

Like terms Multiples of the same combination of algebraic symbols.